METHODS OF GENERATING PLANE CUBIC CURVES.

by

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CONTENTS

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INTRODUCTION.

The theory of plane cubic curves has been treated in detail by Heinrich Schroeter in "Die Theorie Der Ebenen Kurven." For the most part a discussion of the methods of generation which are considered in this paper may be found in the above text. References to other authors and articles will be given from time to time.

A plane cubic curve may be defined as a plane curve which is cut by any straight line in three and only three points.

The purpose of this paper is to give a number of methods by which plane cubic curves may be generated projectively.

I. LOCUS OF VERTICES OF INVOLUTION PENCILS DETERMINED BY INDEPENDENT PAIRS OF POINTS.

Three pairs of points are called independent if they may not be taken as three pairs of opposite vertices of a complete quadrilateral.

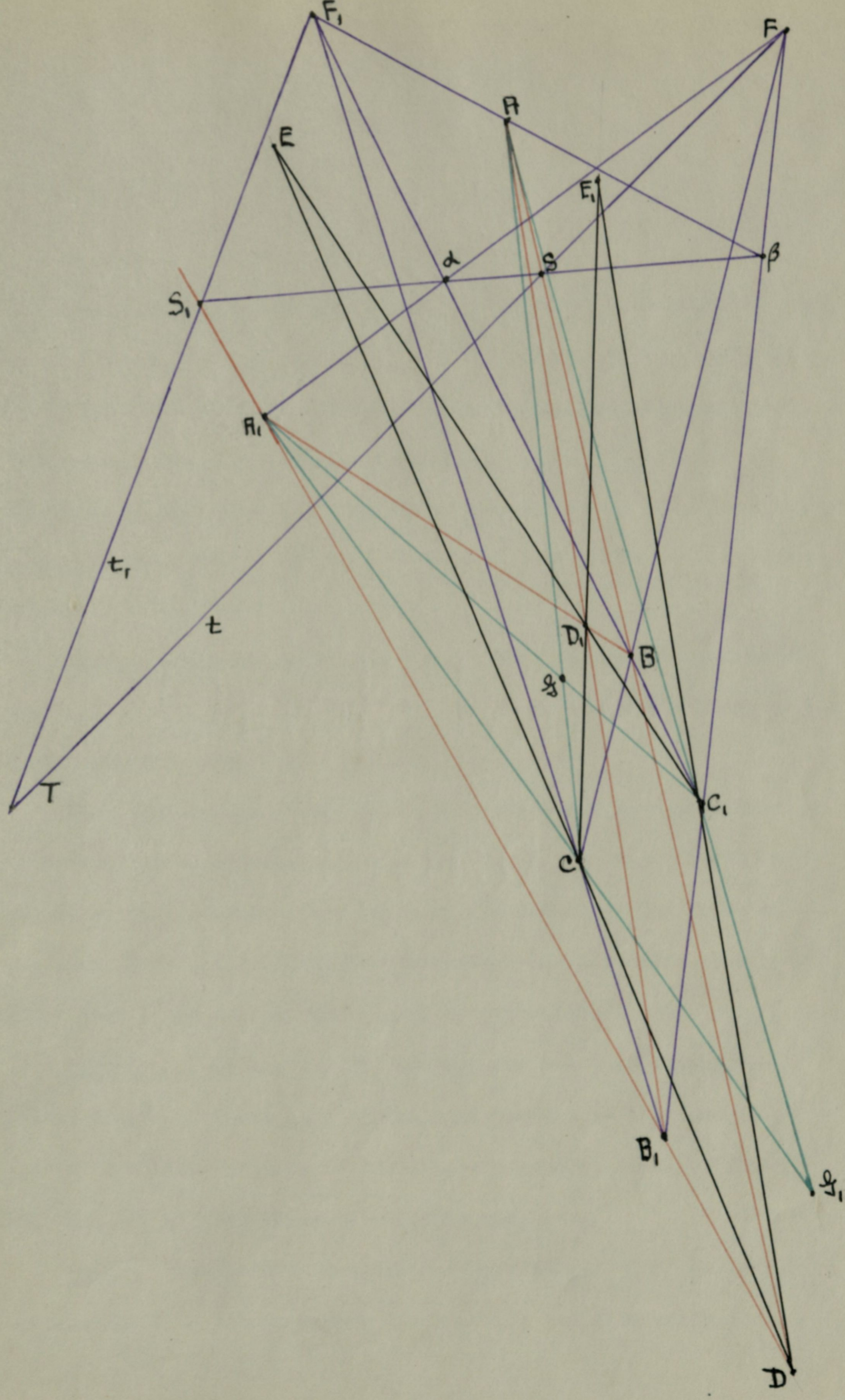
Let AA_1, BB_1, CC_1, be three pairs of independent points. The problem is to find X such that XA, XA_1 ; XB, XB_1 ; Xc, XC_1 ; are pairs of conjugate lines of the same involution pencil.

The points A, A_1, B, B_1, C, C_1, themselves satisfy the demand made upon X. For example let X equal A. Since any two pairs of lines are in involution, AB, AB_1 ; AC, AC_1 ; may be taken as conjugate lines of an involution with A as vertex. A second line through A, which may be called AA, may then be determined such that it will be conjugate to AA_1. In the same way the other five original points will satisfy the condition imposed upon X.

Let the lines AB and A_1B_1 intersect in D; AB_1, A_1B in D_1; BC, B_1C_1 in F; BC_1, B_1C in F_1 ; DC, D_1C_1 in E; DC_1, D_1C in E_1 ; AC, A_1C_1 in G; A_1C, AC_1 in G_1.

The pairs of points AA_1, BB_1, DD_1, are opposite vertices of a complete quadtilateral. By construction $ED \equiv EC$, $ED_1 \equiv EC_1$. Since ED, ED_1 ; EA, EA_1 ; EB, EB_1 ; are conjugate lines of an involution,*

* Any point in the plane may be chosen as vertex of an involution pencil, conjugate lines of which pass thru the ppposite vertices of a complete quadrilateral.


F₁
F
A
E
E₁
α
S
β
S₁
A₁
t₁
t
D₁
B
g
T
C₁
C
B₁
g₁
D

So also are EC, EC_1 ; EA, EA_1 ; EB, EB_1 .

Thus E is a point on our locus. Likewise E_1 is a point of the locus since $E_1 D \equiv E_1 C_1$ and $E_1 D_1 \equiv E_1 C$. Since DA, DA_1 ; DC, DC_1 ; may be taken as pairs of conjugate lines of an involution and $DA \equiv DB$ and $DA_1 \equiv DB_1$, it follows that D is a point of the locus. Similarly D_1 satisfies the conditions placed upon X. In the same way it may be shown that F and F_1 , G and G_1 , are points of the desired locus.

We shall call the pairs AA_1 , BB_1 , CC_1 ;--conjugate points of the locus if XA, XA_1 ; XB, XB_1 ; XC, XC_1 ; are conjugate lines of the same involution when X is a point of the locus.

The construction given above, in which a new pair of points is obtained from any two pairs already found, may be extended to locate as many points of the locus as are desired. The locus of the point X as defined will be represented in this discussion by C^3 . Proof that C^3 is a cubic curve will be given later.

THEOREM I. If AA_1 and BB_1 are any two pairs of conjugate points of C^3 , a third pair of conjugate points is the remaining pair of opposite vertices of the complete quadrilateral of which AA_1 and BB_1 are two pairs of opposite vertices.

PROOF. That points thus obtained are points of C^3 follows immediately from the preceding construction and discussion.

It remains to show that these points are also conjugate points. The pairs CC_1, DD_1, EE_1, are three pairs of opposite vertices of a complete quadrilateral. The involution pencil determined by these points with any point in the plane as vertex must then have as conjugate lines the lines through CC_1, DD_1, EE_1. Since two pairs of conjugate lines determine an involution, the involution with any point X of C^3 as vertex and determined by the three pairs of points AA_1, BB_1, CC_1, must be identical with that determined by X and AA_1, BB_1, DD_1, and by X with AA_1, CC_1, DD_1, and finally by X and CC_1, DD_1, EE_1. Similarly the points FF_1, GG_1, --- may be considered. Thus DD_1, EE_1, FF_1, --- are conjugate points of C^3.

It also follows from the above discussion that any point X of C^3 is the vertex of an involution, conjugate lines of which pass through any pair of conjugate points of C^3. The notation AA_1, BB_1, -- or in general XX_1, will be used to denote pairs of conjugate points of C^3.

<u>DEFINITION.</u> In the involution XA, XA_1 ; XB, XB_1 ; XC, XC_1 ; the line XX conjugate to XX_1 will be called the tangent to C^3 at X.

An involution with vertex X will be denoted by "involution X", an involution with vertex X_1 by "involution X_1."

We shall now give a method of constructing the tangents to C^3. Let $X \equiv F$ and $X_1 \equiv F_1$. Let FA_1 and F_1B intersect in α; FB_1 and

F_1 A in β; $\alpha\beta$ and AB in S; $\alpha\beta$ and A_1 B_1 in S_1 . It follows that $A\beta$ and $B\alpha$ will intersect in F_1 ; $A_1\alpha$ and $B_1\beta$ in F.

By construction $A\alpha$, $B\beta$, F_1 S, are pairs of opposite vertices of a complete quadrilateral, and FA, $F\alpha$; FB, $F\beta$; FF_1 , FS; are conjugate lines of an involution. As $F\alpha \equiv FA_1$, $F\beta \equiv FB_1$, this involution is the one in which we are interested and $FS \equiv t$ is the line conjugate to FF_1 or is tangent to C^3 at F. Similarly F_1 S_1 or t_1 is tangent at F_1 since $A_1\alpha$, A_1 B_1 , $B_1\beta$, $\beta\alpha$, are sides of a complete quadrilateral.

<u>THEOREM II.</u> Tangents to C^3 at a pair of conjugate points intersect at a point on C^3.

<u>PROOF.</u> Let the lines FF_1 and A_1 B_1 intersect in M; t and A_1 B_1 in O; FF_1 and AB in P; t_1 and AB in N; t and t_1 in T.

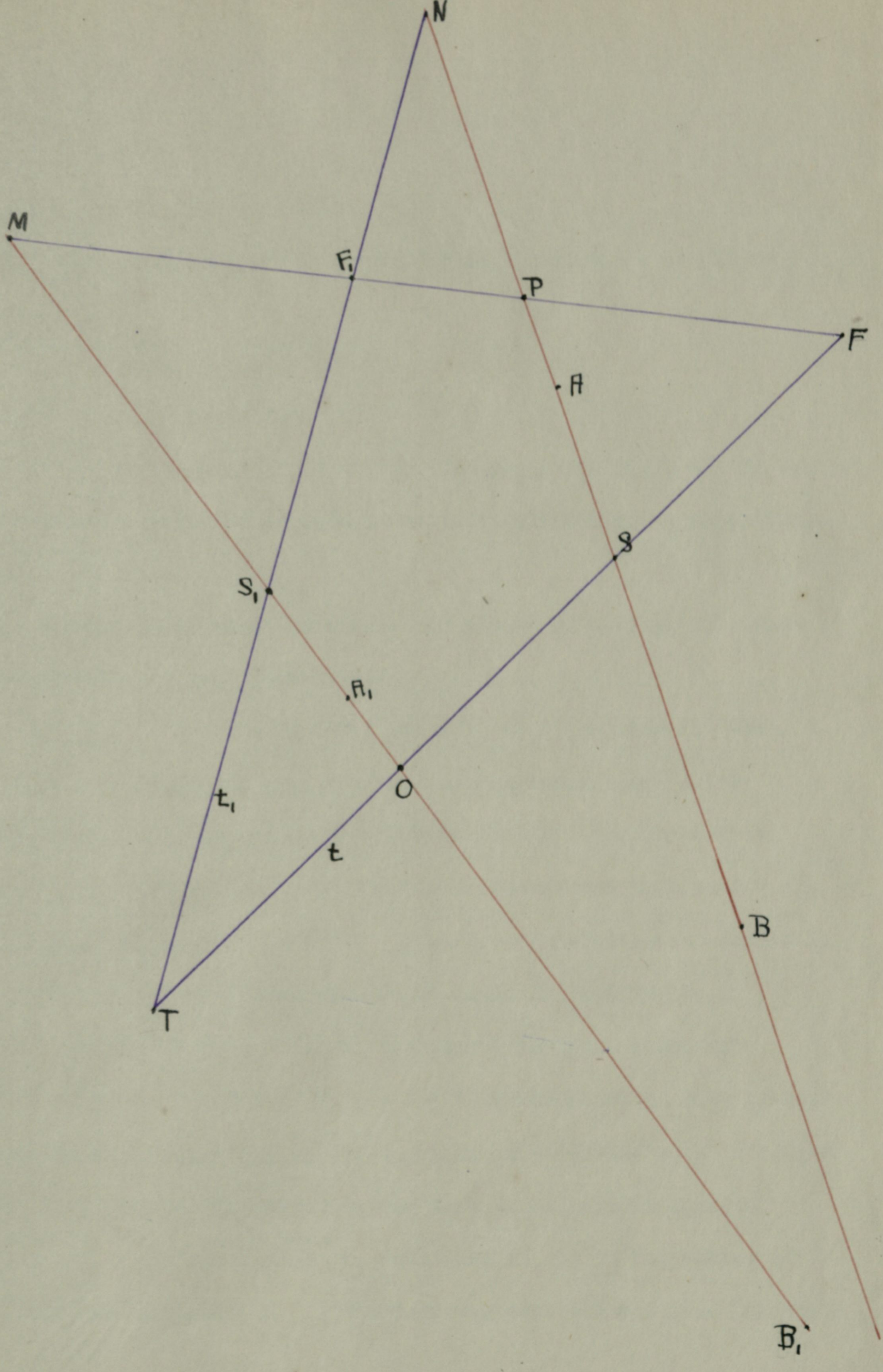
Consider the lines A_1 B_1 and AB and the products of tne three cross ratios formed with the five points A_1 , B_1 , S_1 , M, O and A, B, S, N, P. Since $(A_1 B_1, MO) \cdot (A_1 B_1, OS_1) \cdot (A_1 B_1, S_1M) = 1$, we have,

(1) $F(A_1 B_1, F_1 T) \cdot T(A_1 B_1, FF_1) \cdot F_1(A_1 B_1, TF) = 1.$

Likewise since $(AB, PN) \cdot (AB, NS) \cdot (AB, SP) = 1$, we have,

(2) $F_1(AB, FT) \cdot T(AB, F_1 F) \cdot F(AB, TF_1) = 1.$

Since F and F_1 are conjugate ppints of C^3 and FA, FA_1 ; FB, FB_1 ; —— FT, FF_1 are conjugate lines of involution F, then



N
M
F1
P
F
A
S
S1
A1
O
t1
t
B
T
B1

$F(AB, TF_1) = F(A_1 B_1, F_1 T)$ and

$F_1 (AB, FT) = F_1 (A_1 B_1, TF)$. Substituting these values in (2) we obtain,

(3) $F_1 (A_1 B_1, TF) \cdot T(AB, F_1 F) \cdot F(A_1 B_1, F_1 T) = 1$.

From (1) and (3) it follows that,

$T(A_1 B_1, FF_1) = T(AB, F_1 F)$. Thus TA, TA_1 ; TB, TB_1 ; TF, TF_1 ; are conjugate lines of an involution and T satisfies the conditions for a point on C .

<u>THEOREM III.</u> Lines determined by a pair of conjugate points of C^3 contain a third point of C^3 .

<u>PROOF.</u> If there is a third point T_1 of C^3 on line AA_1 then, $T_1 A$, $T_1 A_1$; $T_1 B$, $T_1 B_1$; --- must be conjugate lines of an involution. Thus AA_1 would be a double line and the involution hyperbolic. It remains to be shown that the second double or invariant line and the vertex T_1 of such an involution can be found.

Let the point of intersection of lines AA_1 and BB_1 be R_1, of AA_1 and CC_1 be R_2. Call R_1' the fourth harmonic point of R_1 with respect to B and B_1, R_2' the fourth harmonic of R_2 with respect to C and C_1. Since the two double lines must divide every conjugate pair harmonically the second double line is definitely located through $R_1' R_2'$. The line $R_1' R_2'$ cuts line AA_1 in T_1 the vertex of the desired involution. In the same way that the point T_1 has been determined, a third point of C^3 may be found on the lines BB_1, CC_1 ---.

The double line $R_1' R_2'$ of involution T_1 contains all of the fourth harmonic points of the R_i' points with respect to the corresponding pair of conjugate points of C^3. In general the double lines of the involutions with vertices the third point of intersection of the lines XX_1 with C^3 must contain all fourth harmonics of the R_i' points with respect to the corresponding pair of conjugate points of C^3.

<u>THEOREM IV.</u> If T is the point of intersection of the tangent t to C^3 at A, and tangent t_1 to C^3 at A_1, and T_1 is the third point of intersection of line AA_1 with C^3, then T and T_1 are conjugate points of C.

<u>PROOF.</u> In the involution A, the pairs of lines AA_1, t; AB, AB_1; AC, AC_1; are conjugate. In involution A_1 the line conjugate to line $A_1 A$ or $A_1 T_1$ is t_1. Thus the point conjugate to T_1 must by definition of conjugate points, be both on t and t_1 or must be the point T.

II. <u>GENERATION OF C^3 THROUGH TWO PROJECTIVE HALF PERSPECTIVE LINE INVOLUTIONS.</u>

<u>Method A.</u>.

Let AA_1, BB_1, CC_1 ---- and in general XX_1 represent as before pairs of conjugate points of C^3. Let line AA_1 intersect line

BB_1 in R_1, and R_1' be the fourth harmonic point of R_1 with respect to B and B_1, also let AA_1 cut CC_1 in R_2, with R_2' as fourth harmonic of R_2 with respect to C and C_1, and finally let AA_1 cut XX_1 in R_i', with R_i' as fourth harmonic of R_i' with respect to X and X_1.

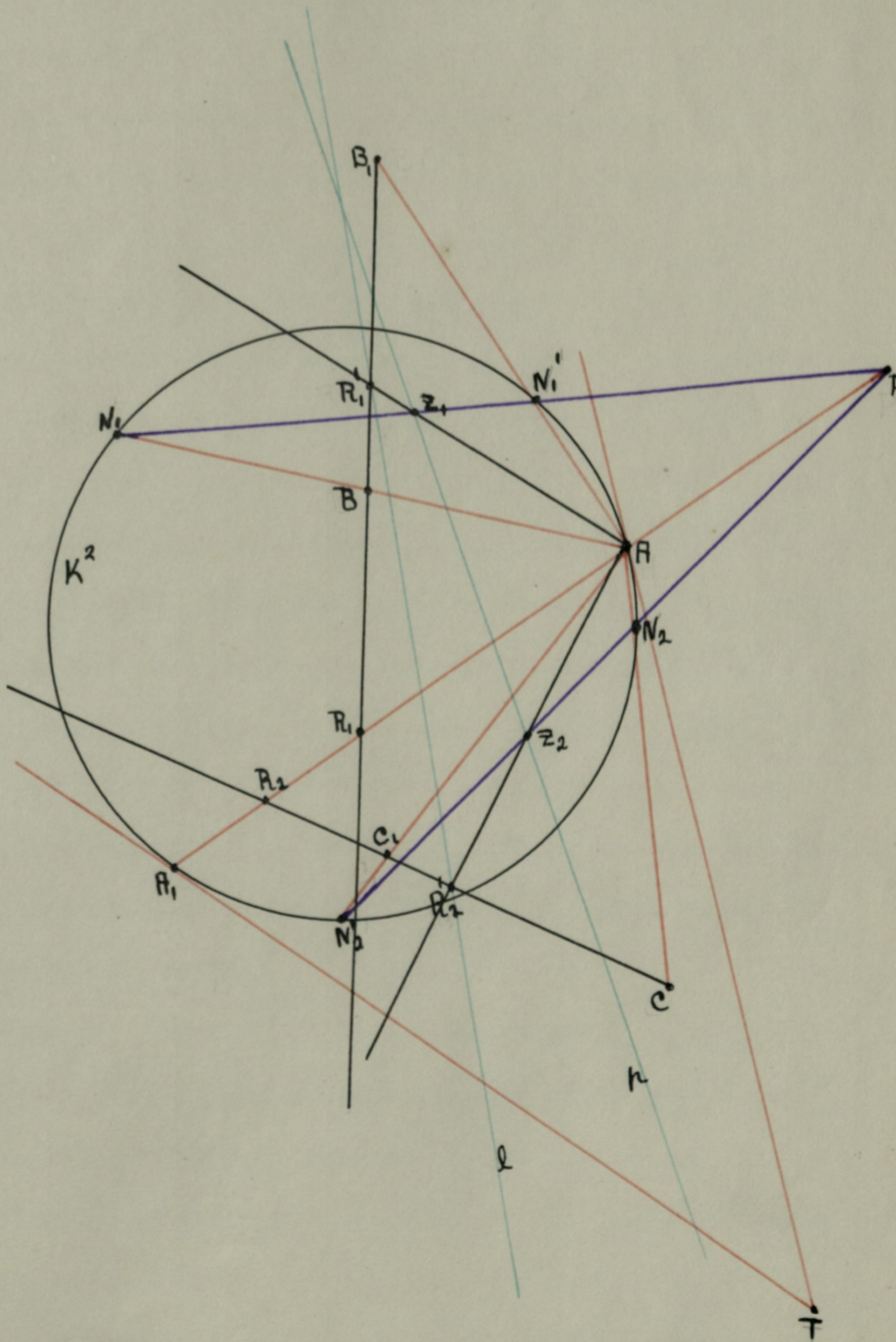
From Section I it follows that R_1', R_2' -- lie on a line l and R' describes a range of points on l. Two perspective pencils of lines are formed by AR' and A_1R'.

Through A construct a conic K^2 tangent to the line AT where T represents the third point of intersection of the tangent to C^3 at A with our locus. If lines AB, AB_1; AC, AC_1; -- cut K^2 in N_1, N_1'; N_2, N_2'; -- the lines NN' pass through one point P. *

Point P must be on AA_1 for in involution A lines AA_1 and AT are conjugate. The second point of intersection of AT with K^2 is A, of AA_1 with K^2 is some point Y on AA_1. The line connecting these points is AA_1 which must therefore pass through P.

Let p the polar of P with respect to K^2 cut line N_1N_1' in Z_1. By definition of the polar of a point with respect to a conic we then have a harmonic set $H(PZ, N_1 N_1')$ and therefore $A(PZ_1, N_1 N_1')$ are four harmonic lines.

*(If DD', CC', EE', -- are pairs of conjugate points in an involution on a conic the lines DD', CC', EE' are on a point.)



B_1
N_1'
P
R_1
Z_1
N_1
B
A
K^2
N_2
R_1
Z_2
R_2
C_1
A_1
R_2
N_2
C
p
l
T

Thus $A(A_1 Z_1, BB_1)$ are four harmonic lines and, since $A(A_1 R_i',$ $XX_1)$ by construction form a harmonic pencil, we have lines $AR_i' \equiv$ AZ_i. It follows that $P(N_1 N_1', N_2 N_2', ---) \overline{\overline{\wedge}} A(R_1', R_2', R_3' ---)$ since each line of the first intersects a line of the second on the line p. Every line of the pencil of lines at P furnishes a definite pair of conjugate lines in involution A. This Schroeter expresses as reducing the line involution by a pencil of lines at P and defines two line involutions as projective if the pencils of lines by which they can be reduced are projective. Thus, the two line involutions A and A_1 are projective since their respective reduction pencils will be perspective with the pencils $A(R_1' R_2' --)$ and $A_1(R_1' R_2' --)$ which in turn are perspective with each other.

Corresponding line pairs of the two involutions A and A_1 cut necessarily in X and X_1 and also in two more points S, S_1, which are conjugate points of C^3, (Theorem I), Since in each case we have a complete quadrilateral formed by two pairs of corresponding lines of the two involutions.

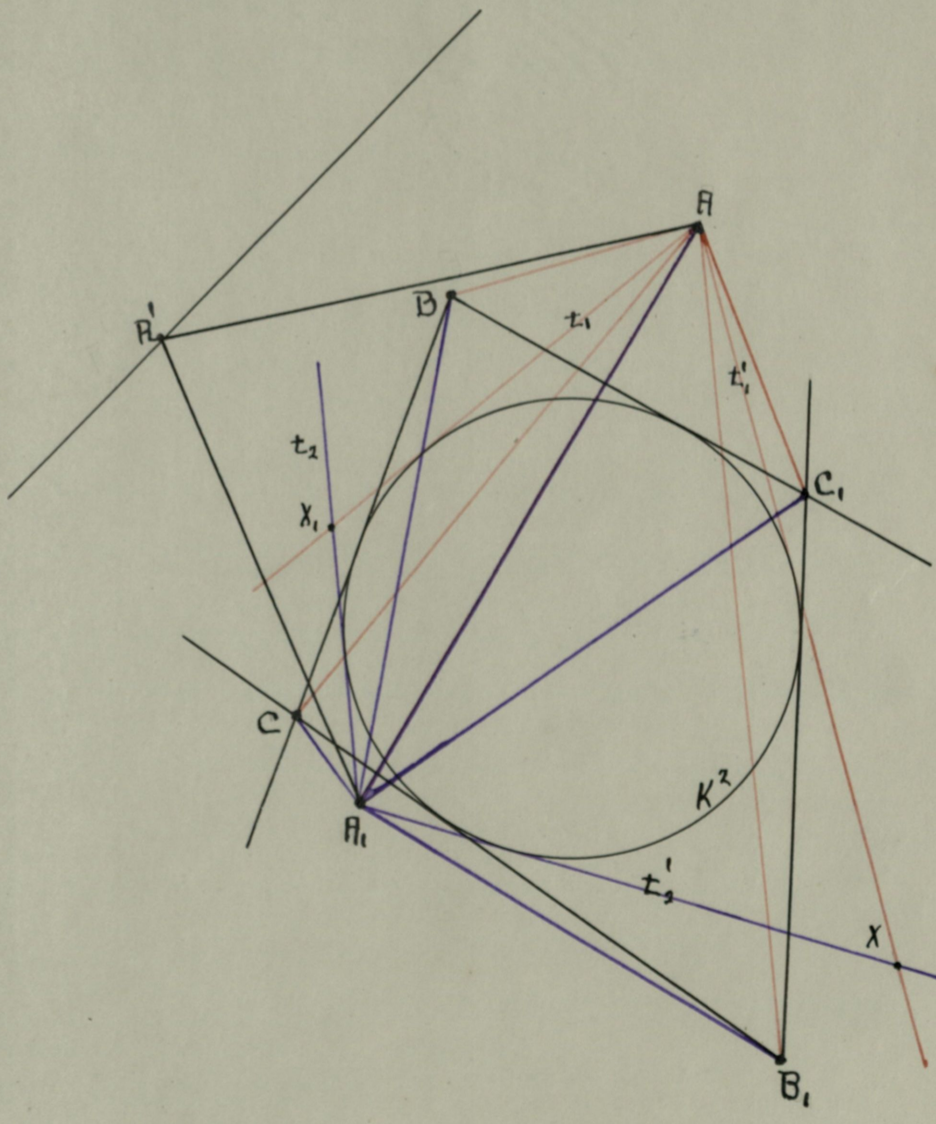
The lines AA_1, AT of involution A correspond to $A_1 A$, $A_1 T$ of involution A_1. When a line connecting the vertices of two projective involution pencils, considered as a line of the one corresponds to itself considered as a line of the other, Schroeter defines this as "Half Perspective" position.

The locus of C^3 will thus be the vertices and points of intersection of corresponding lines of two projective half perspective line involutions. Two pairs of corresponding lines give us each time two pairs of conjugate points. The infinite number of common points on the line connecting the two vertices are not to be included in the locus. The determination of the third point on this line was given in Section I.

In the construction given in (I) as many points as desired could be obtained but these points were not continuous. By the above method we may take the conic K^2 tangent at A and A_1 to the lines AT and A_1T and as the point R' moves continuously on the line l we obtain corresponding line pairs and by their points of intersection continuous points on our cubic.

METHOD B.

Again let AA_1, BB_1, CC_1 be three pairs of conjugate points of C^3. Form a complete quadrilateral with the lines BC, BC_1, B_1C, B_1C_1, as sides and consider the pencil of conics tangent to these four lines. All pairs of tangents from one fixed point to such system of conics are conjugate lines of an involution of which the lines drawn to the opposite vertices of the quadrilateral are also conjugate lines. Call one conic of the pencil K^2. Let the tangents from A to K be t_1 and t_1'; from A_1 to K^2 be t_2 and t_2'. Thus AB, AB_1; AC, AC_1; t_1 t_1'; and A_1B, A_1B_1; A_1C, A_1C_1; $t_2 t_2'$; are pairs of conjugate lines of involutions



A
A'
B
t_1
t_1'
t_2
X_1
C_1
C
A_1
K^2
t_2'
X
B_1

A and A_1. Corresponding to each conic of the pencil will be two definite pairs of conjugate lines of the two involutions.

THEOREM V.

If AA_1, BB_1, CC_1, are conjugate points of C^3, the involutions A and A_1, determined by tangents from A and A_1 to conics of the pencil with the four common tangents BC, BC_1, B_1C, B_1C_1, are projective.

PROOF.

The poles R' of line AA_1 in reference to all conics of the pencil lie on a line a', and may be found by connecting the poles of two conics with respect to line AA_1. *

For every point R' a definite conic of the pencil having R' as pole of line AA_1 is determined. **

If R' is the pole of line AA_1 in regard to K^2 then t_1, t_1' are harmonically divided by lines AA_1 and AR'. As K^2 varies and R' describes a range of points on a' we have associated with every pair of conjugate lines of involution A, the pair of lines AR', AA_1 which divide the above pair harmonically.

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* (If a pair of lines are conjugate lines with regard to any two conics of a family having four common tangents, they are conjugate lines with regard to every conic of the family.)

** ("The Principles of Projective Geometry" by Hatton. Page 243.)

Similarly with involution A_1 we have associated the pencil of lines $A_1(R')$ each line of which is the fourth harmonic of $A_1 A$ with respect to a definite pair of conjugate lines of the involution. Moreover $A(R') \overline{\wedge} A_1(R')$. We thus have involution A projective with involution A_1 since it was shown in Method A of this section that two line involutions A and A_1 are projective if they are so situated that the pencil of lines formed by the harmonic conjugates of line AA_1 with respect to each pair of conjugate lines of involution A, is projective with the pencil of lines similarly formed from involution A_1.

Involutions A and A_1 are also in half perspective. Among the conics of the pencil only one is tangent to the line AA_1 and since corresponding pairs of lines of the two involutions are tangents from A and A_1 to the same conic, we have in this case AA_1, t_1 of involution A, corresponding to $A_1 A$, t_2 of involution A_1. This satisfies our definition for half perspective position.

In Method A, it was shown that the locus of the points of intersection of corresponding lines of two projective half perspective line involutions is a C^3 which passes through the vertices of the two involutions. We may thus state the following:

<u>THEOREM VI</u>.

Given a pencil of conics with four common tangents, the locus of the points of intersection of two pairs of tangents from two fixed points to each conic is a C^3 passing through the two fixed points and the six points of intersection of the four common tangents.

The following theorem might also be noted here.

THEOREM VII

Given any four pairs of conjugate points AA_1, BB_1, CC_1, DD_1, of C^3, the eight lines AB, AB_1, A_1B, A_1B_1, CD, CD_1, C_1D, C_1D_1, will be tangent to the same conic.

PROOF.

Consider the conic K^2 which has the five tangents CD, CD_1, C_1D, C_1D_1, AB. The first four tangents will determine a pencil of conics. Since AB is tangent to K^2 the second tangent to K^2 from A must pass through the conjugate of B with respect to C^3 or through B_1.

Consider now the involutions B and B_1. Conjugate lines of these involutions must be tangent to the same conic. The conjugate to B_1A will be the second tangent to K^2 from B_1, and since A_1 is conjugate to A with respect to C^3, this second tangent must contain A_1. Similarly the second tangent to K^2 from B must pass through A_1 or the two tangents from B and B_1 intersect at A_1 and A. Thus we have the five given tangents and in addition the tangents AB_1, A_1B, A_1B_1, or in all eight tangents to the same conic.

III. PROOF THAT C^3 IS A CUBIC.

THEOREM VIII.

Any line cuts C^3 in three and only three points.

PROOF.

Consider C^3 as generated by two projective half perspective line involutions. Let g be any line in the plane cutting the two generating involutions A and A_1.

Using the notation of the preceding sections AA_1, BB_1, CC_1 --- are pairs of corresponding points on C^3 and AA_1, AT; A_1A, A_1T; are pairs of corresponding conjugate lines. If line AA_1 cuts g in point O, consider a conic K^2 tangent to g in O. Let lines AB and g intersect in R_1; AB_1 and g in R_1'; AC and g in R_2; AC_1 and g in R_2'; ----------; A_1B and g in N_1; A_1B_1 and g in N_1'; A_1C and g in N_2; A_1C_1 and g in N_2'; -------.

From any pair of points R_1 R_1' draw the two remaining tangents r_1 r_1' to K^2 and let r_1 r_1' intersect in a point P_1. As R and R' vary over g the point P moves in a straight line l. *
This point range is projective with the pencil of lines formed by the polars of the points P_1, P_2 -------. **

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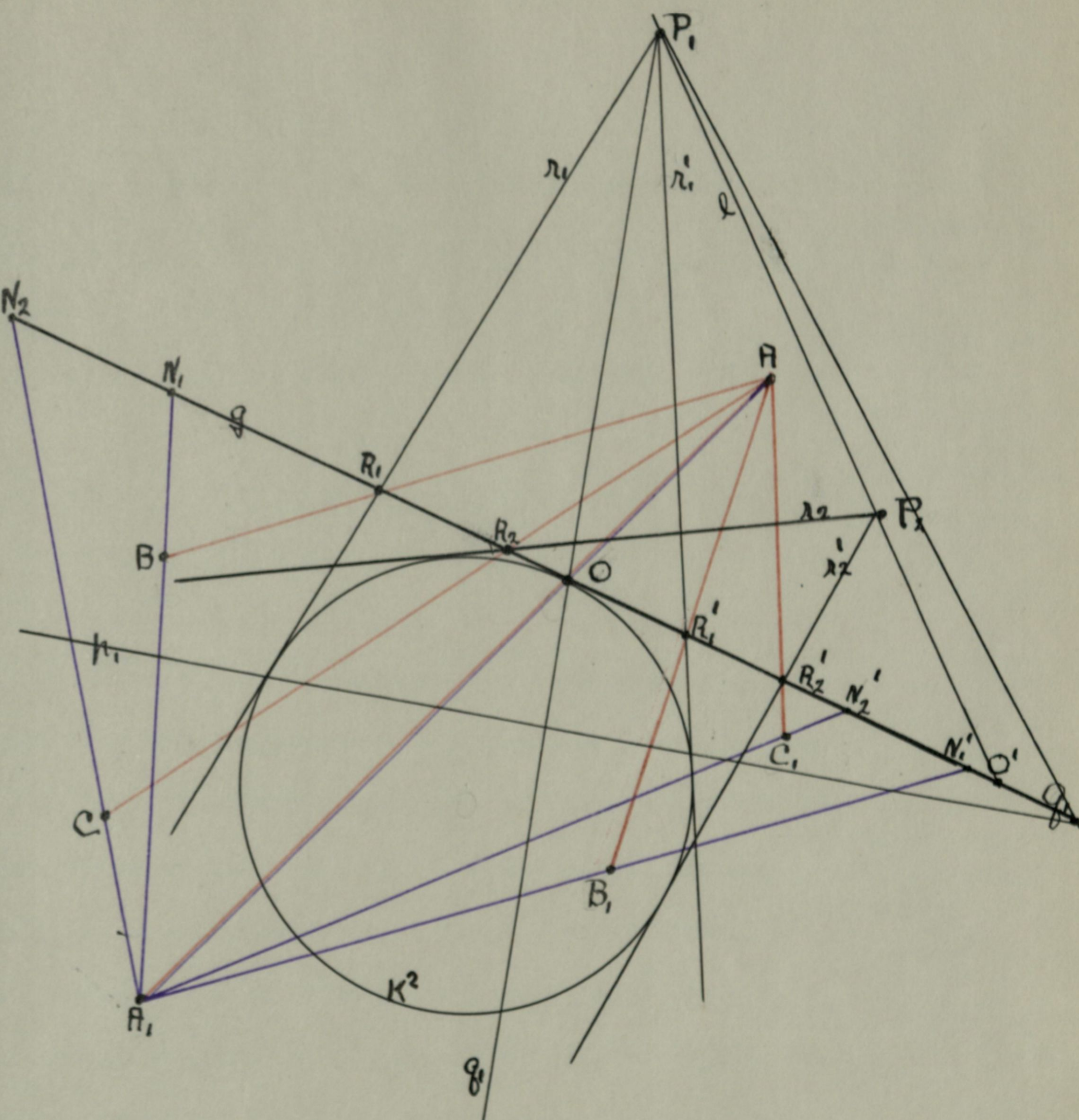
*(The locus of points of intersection of tangents to a conic from pairs of corresponding points of an involution on a tangent to a conic, is a straight line.)

**(As a point P varies over a pencil of points its polar with respect to any conic varies over a projective pencil of lines.)



P₁
r₁
r₁'
l
N₂
N₁
g
A
R₁
R₂
r₂
P₂
B
O
r₂'
p₁
R₁'
R₂'
N₂'
C₁
N₁'
O'
q
C
B₁
A₁
K²
q₁

The range of points P_1, P_2--- is also projective with the range of points on g formed by the fourth harmonic points of O with respect to the pairs of conjugate points $R_1 R_1'$, $R_2 R_2'$---. To prove this consider the point P_1 whose polar cuts g in some point Q_1. If the polar of P_1 passes through Q_1 the polar of Q_1 passes through P_1. The polar of Q_1 however, passes through the points of contact of the tangents to K^2 from Q_1. One of these points of contact is O, so the polar of Q_1 is $P_1 O$. Thus lines $P_1 O$ and $P_1 Q_1$ are harmonic conjugates with respect to r_1 r_1' and the polars of P with respect to K^2 cut g in the fourth harmonic points Q of O with respect to the pairs of conjugate points RR'. Thus the ranges of points ($P_1 P_2$ ---) and (Q_1 Q_2 --) are projective since each range is projective with the pencil of polars. Similarly, we may reduce the involution $N_1 N_1'$, $N_2 N_2'$, ---- to a range of points ($P_1' P_2'$ ----) on a definite line l'.

The ranges of points (P_1 P_2---) and (P_1', P_2' ----) are projective. For, from the preceding we have (P_1, P_2 --) $\overline{\wedge}$ (Q_1, Q_2 ---) and (P_1', P_2' --) $\overline{\wedge}$ (Q_1', Q_2' --), if Q' represents the point of intersection of g with the polar of any point P'. In Section II the pencil of lines A(Q) was proved perspective with the pencil A(Q'). Thus (Q_1, Q_2 --) $\overline{\wedge}$ (Q_1', Q_2' ---) and (P_1, P_2 ---) will be projective with (P_1', P_2' --).

The lines AA_1 and g intersect in O; AT and g in O'; A_1A and g in O; A_1T and g in O_1'.

In the two involutions on g determined by its intersections with involutions A and A_1, we thus have as conjugate points O, O' and O, O_1' respectively. The tangents to K^2 from O and O' must intersect in a point P of line l, and from O and O_1' in a point P' of l'. Since the tangent to K^2 from O is the line g it follows that l cuts g in O' and l' cuts g in O_1'.

The lines $P_1 P_1'$, $P_2 P_2'$,--- envelope a conic K_1^2. *
The line $O'O_1'$ or g will be tangent to K_1^2 so the two conics K^2 and K_1^2 have a common tangent g. In general two conics have four common tangents. Since one of these is real at least one other must be real while the two remaining may be imaginary. We are interested in the three remaining common tangents for every time one of the tangent pair from P_i to K^2 and P_i' to K^2 are united in a line $P_i P_i'$ (tangent to K_1^2) we have a point of the point pair R_i, R_i' coinciding with a point of the corresponding pair N_i, N_i'. This means that we have a point of intersection of g with C^3.

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*(The envelope of the lines joining corresponding points of two projective ranges is a conic touching the bases of the ranges.)

Thus any line g contains in general three points of C^3 of which one is real and the remaining two may be either real or conjugate imaginary. Only because of the half perspective position of the two generating line involutions does the number of points on g break down from four to three.

IV. (A) GENERATION OF C^3 THRU TWO PENCILS OF CONICS.

(B) CERTAIN RELATED FACTS CONCERNING PENCILS OF CONICS.

(A) The C^3 will be obtained as in Section I as the locus of a point X which is the vertex of an involution pencil, conjugate lines of which pass through three pairs of independent points, AA_1, BB_1, CC_1.

Since the anharmonic ratio of any four points of an involution is equal to that of their four conjugates, $X(BC, AA_1) = X(B_1C_1, A_1A)$.

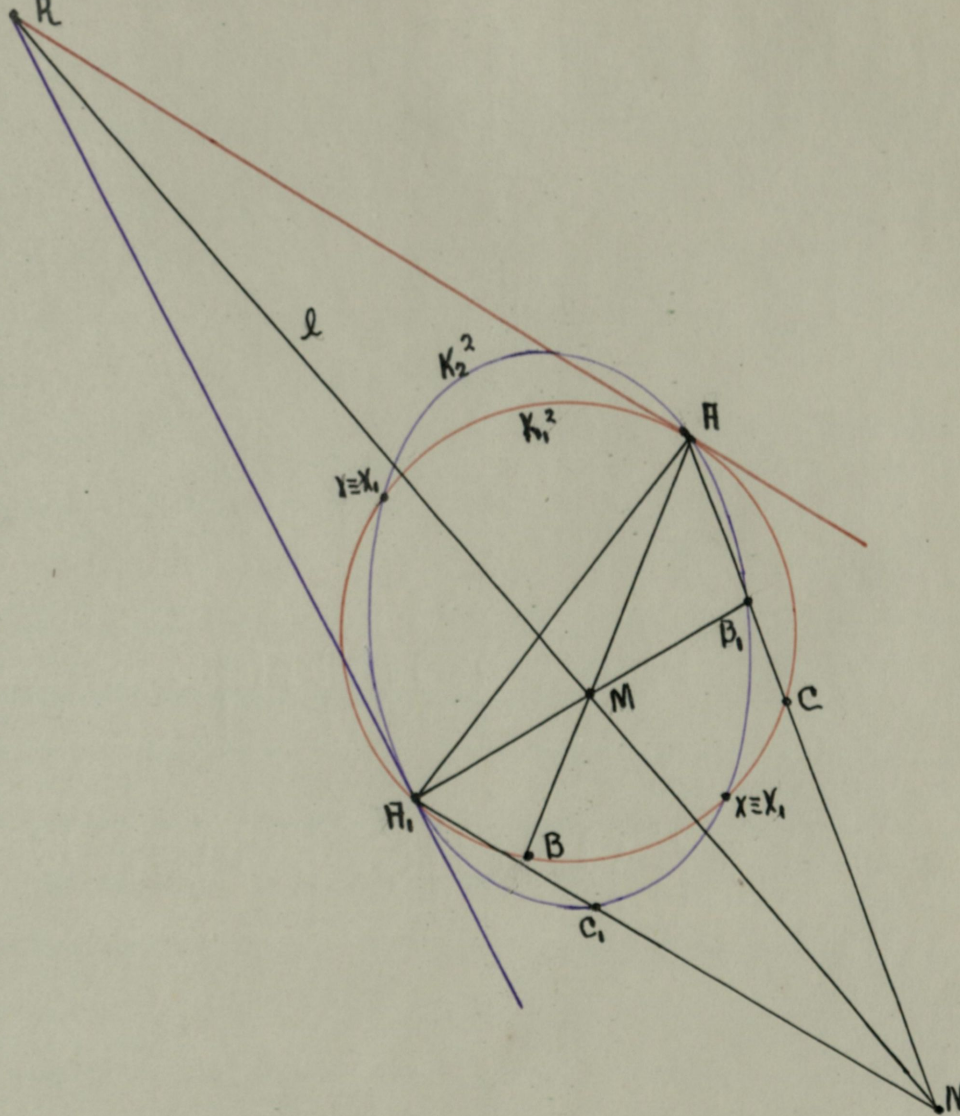
The locus of X for which the anharmonic ratio of the lines joining X to four fixed points is constant, is a conic, passing through the four given points. Let $X(A_1A, BC) = x$ and construct a line t_1 through A_1 such that the value of the anharmonic ratio of the four lines t_1, A_1A, A_1B, A_1C, is equal to x. Thus the conic which is the locus of X is completely determined through A, A_1, B, C, and the tangent t_1 at A_1. However, $x = X(AA_1, B_1C_1)$, so X may lie on a second conic through A, A_1, B_1, C_1. As these two conics have A and A_1 in common the remaining two points of intersection satisfy our demand for X.

By taking new values for x we may obtain as many positions of X as are desired.

To obtain consecutive points on C^3 we may proceed in the following manner. Let lines AB cut $A_1 B_1$ in M and AC cut $A_1 C_1$ in N. Let the line MN be l and let R be any point on l. By construction $A(A_1 B, C R) = A_1 (AB_1, C_1 R)$. Place a conic K_1^2 through A, A_1, B, C and tangent to line AR and a conic K_2^2 through A_1, A, B_1, C_1 and tangent to line $A_1 R$.

For any position of X on K_1^2, $X(BC, A_1 A) \equiv A(BC, A_1 R)$ since the anharmonic ratio is constant as X moves over K_1^2, and since lines $XA \equiv AR$ when X coincides with A. Similarly for any position of X_1 on K_2^2, $X_1 (B_1 C_1, AA_1) \equiv A_1 (B_1 C_1, AR)$. For points of intersection of K_1^2 and K_2^2, $X \equiv X_1$ and $X(BC, A_1 A) \equiv X_1 (B_1 C_1, AA_1)$ or $X(BC, A_1 A) \equiv X(B_1 C_1, AA_1)$ since $A(BC, A_1 R) \equiv A_1 (B_1 C_1, AR)$. Thus points of intersection of K_1^2 and K_2^2 other than A and A_1 satisfy the demand made upon points of C^3.

Let R move on l causing K_1^2 and K_2^2 to describe two pencils of conics through four common points A, B, C, A_1 and A_1 B_1, C_1 A. The points of intersection of corresponding conics of the two pencils will generate a cubic curve and a straight line. The straight line appears when R is on line AA_1. Then not only are A and A_1 on both conics but the line AA_1 is tangent to each.


R
l
K₂²
K₁²
A
X≡X₁
B₁
M
C
A₁
X≡X₁
B
C₁
N

Thus our conics have degenerated into two intersecting lines AA_1, BB. The two conics have the line AA_1 in common, and, as in the case of the two projective half perspective involutions, we must exclude from our locus the infinite number of points of intersection on the one resulting straight line.

(B) Pencils of conics. The polars of a point P with regard to the pencil of conics through four fixed points A, A_1, B, C, move through a fixed point P_1 and similarly the polars of a second point Q move through a point Q_1. It will be shown that the two pencils of polars through P_1 and Q_1 are projective. The pole of the line PQ with respect to one conic K_1^2 of the pencil lies at the intersection of the polars of P and Q with respect to K_1^2. *
Similarly for conic K_2^2 of the pencil the above is true, so the points of intersection of the corresponding lines of the pencils of polars through P_1 and Q_1 are in each case the poles of the line PQ with respect to the conics. The locus of the poles of the line PQ in regard to the conics of the pencil is a conic passing through P_1 and Q_1. **

---

*(If the polar of A passes thru B the polar of B passes thru A.)

**(If P is a variable point on a fixed line l and if P' is the point conjugate to P with respect to two therefore all conics of a pencil of conics thru four fixed points A, B, C, D, then the locus of P' is a conic S which passes thru the poles of l with respect to the conics of the pencil.)

Thus our two pencils of polars thru P_1 and Q_1 are projective since the corresponding lines intersect on a conic thru P_1 and Q_1.
The above holds for any points P and Q or for any two positions of a point P. Thus we may state the following theorem,

THEOREM IX.

If a pencil of conics is taken thru four points the pencils of polars with regard to these conics for any two positions of a point P are projective.

In case P coincides with one of the four fixed points of the pencil of conics the polars become a pencil of tangents at this point. Two pencils of conics are defined as projective if the pencil of tangents at one of the fixed points of the first pencil is projective with a pencil of tangents at a fixed point of the second pencil, or in other words, a pencil of conics is defined as projective with the pencil of tangents at one of the fixed points of the pencil.

THEOREM X.

If a line g thru one of the four fixed points of a pencil of conics cuts the conics a second time in R_1, R_2 ---, the range of points R_1, R_2 --- is projective with the pencil of polars of any point P with regard to the pencil of conics, and therefore projective with the pencil of conics.

PROOF.

Consider the pencil of conics thru A, B, C, D, and the line g thru A cutting the conics also in R_1, R_2, R_3, ---. Choose any point P on g and take N_1 as the fourth harmonic of P with respect to A and R_1. $(AR_1, PN_1) = -1$ and thus $(AP, R_i N_i)$ is constant for any position of R and the corresponding position of N. As the conics vary N describes a range of points on g perspective with the pencil of polars of the point P on g. With respect to each conic, the polar of P will cut g in the fourth harmonic point of P with respect to the two cutting points of g with the conic, or the pencil of polars of P will cut g in N_1, N_2--- and all pencils of polars with respect to this pencil of conics will be projective with the range of points N_1, N_2--. Moreover $(R_1, R_2 ---) \,\bar{\wedge}\, (N_1, N_2 --)$, * and it follows that the range of points R_1, R_2 --- is projective with the pencil of polars for any point with regard to the pencil of conics. Since the pencil of polars is projective with the pencil of conics the range R_1, R_2-- is in addition projective with the pencil of conics.

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*(If four collinear points A, B, P, Q are such that (AB, PQ) is constant and A and B are fixed, then the ranges described by P and Q are projective.)

V. CONIC WEBS, CONIC NETS, AND POINT TRIPLES.

Two conics with four common tangents determine a family of conics tangent to the same four lines. Let A^2, B^2, C^2 be three conics not belonging to the same family. Grouping them by pairs we can form three new families. If A_1^2, B_1^2, C_1^2 are any three conics, one taken from each of these families, we may by again grouping in pairs obtain three more families. This process may be continued indefinitely and all of the conics thus obtained form a conic Web.

If we consider a family of conics with four common points in the place of four common tangents and build a parallel system from three conics A^2, B^2, C^2 not belonging to the same family, the resulting system is a conic net.

A conic of either a net or a web will be said to be determined by two pairs of conjugate points AA_1 and BB_1 of C^3, having reference to the four points themselves if considering the net and the four connecting lines AB, AB_1, A_1B, A_1B_1, if discussing the web.

THEOREM XI.

Tangents from a point X of C^3 to all conics of a web determined by three pairs of conjugate points of C^3, are conjugate lines of the involution pencil determined by X and any two pairs of conjugate points of C^3.

PROOF.

Take three pairs of conjugate points of C^3, AA_1, BB_1, CC_1. Let A^2 be a conic of the family of conics determined by the four common tangents BC, BC_1, B_1C, B_1C_1; C^2 a conic determined by AB, AB_1, A_1B, A_1B_1, and B^2, a conic by AC, AC_1, A_1C, A_1C_1.

If X is a point of C^3 it is the vertex of an involution pencil conjugate lines of which are determined by the points A, A_1; B, B_1, C, C_1. Moreover, the tangents to A^2 from X are conjugate lines of the involution determined by X and the points B, B_1; C, C_1, the tangents to B^2 similarly belong to the involution determined by X and A, A_1; C, C_1, and the tangents to C^2 belong to that determined by A, A_1; B, B_1. It follows that the tangents to A^2, B^2, C^2 from X of C^3 belong to the same involution and since all conics of the web are obtained from families determined by these in pairs, the tangents from X to all conics of the web belong to the involution determined by X and any two pairs of conjugate points of C^3.

THEOREM XII.

The poles of a line x in regard to all conics of a web lie on a line x' and if x is a line thru a pair of conjugate points of C^3 (the web being determined by three pairs of conjugate points of C^3) then x' cuts x in a point of C^3.

PROOF.

The poles of a line x in regard to all conics of a family lie on a line x' and the poles of x' lie on x. Let XX_1, BB_1, CC_1, be conjugate points of C^3 and the line XX_1 be represented by x. Among the conics tangent to the lines BC, BC_1, B_1C, B_1C_1, are two conics which have degenerated into the points pairs B, B_1 and C, C_1. The pole of x with regard to conic B, B_1 will lie on the line BB_1 and be the fourth harmonic of the intersection of x with BB_1 with respect to B and B_1. The pole of x with regard to conic C, C_1 is the fourth harmonic of the intersection of x with line CC_1 with respect to C and C_1.

Call the line thru the two poles x'. This line contains the poles of x with regard to all conics of the family. Moreover, in Section I in the proof that any line thru a pair of conjugate points XX_1 cuts C^3 in a third point, this line x' (called $R_1' R_2'$) containing the fourth harmonic points of the line XX_1 with respect to any pair of conjugate points of C^3, was shown to cut XX_1 in its third point of intersection with C^3.

It remains to show that this line x' also contains the poles of x with regard to the conics of the web. The harmonic conjugates of one pair of an involution are also harmonic conjugates of every other pair. Connect any pole P of x with regard to a conic of the family to the point X of the line x. This line and x are harmonic conjugates with respect to the tangents from X to that conic. Since the tangents from X to all conics of the web belong to the same involution, lines XX_1 or x and XP, are harmonic conjugates with respect to the pairs of tangents from X to all conics of the web. The same statements may be made for X_1 and it follows that x' or the locus of P must contain the fourth harmonic of x with respect to the tangents to the conics of the web from X or X_1 on x. * Thus x' is the locus of the poles of x with regard to the conics of the web.

In a dual way the following may be stated:

THEOREM XIII.

The polars of a point X of C^3 with regard to the conics of the family and also the conics of a net determined by three pairs of conjugate points of C^3, lie on a point X_1 .

These are conjugate points with reference to the pencils of conics and the involutions so determined, and thus are conjugate points of C^3, or the following may be stated:

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*(If from a variable point on a straight line s, pairs of tangents a and a' are drawn to a conic the envelope of the harmonic conjugates of s with regard to a and a' is a fixed point S. The point S is called the pole of s.)

THEOREM XIV.

If the polars of a point X with regard to the conics of the family and also the conics of a net determined by three pairs of conjugate points of C^3 lie on a point X_1, then X and X_1 are conjugate points of C^3.

If AA_1, BB_1 are opposite vertices of a complete quadrangle, the diagonal triangle is self polar with respect to all conics thru A, A_1, B, B_1.

DEFINITION.

The vertices of a common self polar triangle for any two conics of a net, and thus for all conics of the determined pencil, will be called a point triple.

The pencil of conics thru A, A_1, B, B_1 shall be represented by $[A^2, B^2]$.

Consider the net of conics determined by three pairs of conjugate points AA_1, BB_1, CC_1, and a point triple of a particular pencil of the net. If M is one vertex of the triple the polars of M with respect to all conics of the pencil must coincide in one line NO. The polar of M with regard to a conic K^2 of the net not belonging to this pencil will cut the line NO in some point M_1. The polars of all conics of the pencil determined by any conic of the first pencil and the conic K^2

will then pass thru the point M_1, and since the net is determined by three conics not belonging to the same pencil, it follows that the polars of M with respect to all conics of the net pass thru one point M_1. From Theorem 14 the point M must then be a point of C^3.

If P is a point of C^3 the polars of P with regard to the conics $[B^2, C^2]$ go thru a fixed point P_1 which is a conjugate point of C^3 with respect to P. The polars thru P_1 are projective with the pencil of conics. (Theorem 9 and definition of projective pencil of conics.) Thus to each line thru P_1 there is one definite conic of the pencil. Let g be a line thru P_1 and call the conic A_1^2 of the pencil $[B^2, C^2]$ which has P and g as pole and polar. Similarly we can determine a conic B_1^2 of the pencil $[A^2, C^2]$ for which P and g are pole and polar. The two pencils $[B^2, C^2]$, $[A^2, C^2]$ belong to the net determined by the three pairs of conjugate points AA_1, BB_1, CC_1, of C^3. Since the conics A_1^2 and B_1^2 have the same pole and polar all conics of the pencil determined by A_1^2 and B_1^2 will have P and g as pole and polar.

If P is chosen as one point of a point triple for a known pencil of conics of the net, the other two points must lie on g.

Since the two remaining points must also lie on C^3 they must be the second and third points of intersection of g with C^3. (P_1 being the first cutting point of g with C^3). Since any line g thru P_1 might have been chosen, it follows that any point Q of C^3 may be chosen as the second vertex of our triple. The third vertex is then completely determined.

Thus if any two points P and Q of C^3 have been chosen as vertices of a triple, connect Q with P_1, the conjugate point of P, and the third point of intersection of the line QP_1 with C^3 is the third triple point R. Since the third vertex is fixed it must follow that the line PQ_1 will cut C^3 in R, or P_1Q cuts PQ_1 in R, a point of C^3. The conjugate to R will be R_1 the point of intersection of PQ and P_1Q_1. (Theorem I). Thus P_1, Q_1, R_1 lie in a straight line, and we may state:

THEOREM XV.

The conjugate points of the vertices of a point triple of C^3 lie on a line and are the points of intersection with C^3 of the three sides of the triangle formed by the vertices of the triple.

The converse may also be stated.

THEOREM XVI.

If a line cuts C^3 in three points P_1, Q_1, R_1, the three conjugate points P, Q, R form a point triple.

PROOF.

P and Q may be chosen as two vertices of a triple. The third vertex R' is the point of intersection of lines PQ_1 and $P_1 Q$. From Theroem I it follows that PQ cuts $P_1 Q_1$ in a point of C^3 which by hypothesis must be R_1, and moreover if PQ cuts $P_1 Q_1$ in R_1, PQ_1 cuts $P_1 Q$ in R, conjugate to R_1. Thus R' and R coincide, or P,Q,R are vertices of a triple. We have then with three pairs of conjugate points PP_1, QQ_1, RR_1, four lines PQ_1R, PQR_1, P_1QR, $P_1Q_1R_1$, and four point triples P_1QR_1, P_1Q_1R, PQ_1R_1, PQR.

THEOREM XVII.

The six points of two point triples of C^3 always lie on a conic.

PROOF.

Two pencils of a net (by definition of a net) must have one conic in common and two self polar triangles for the conic have their six vertices on a second conic. *

THEOREM XVIII.

A conic thru three vertices of a point triple of C^3 cuts C^3 in three new points of a triple.

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*(Two triangles which are self polar with respect to the same conic have their six vertices on a second conic and, their six sides tangent to a third conic.)

PROOF.

Any conic cuts C^3 in six points. Let conic K^2 cut C^3 in the three points P, Q, R of a point triple and in addition in the points M, N, O. We may choose M and N as two points of a third triple. By the preceding theorem there is a conic thru P, Q, R and M, N, O' if O' is the third vertex of the triple determined by M and N. Since there is only one conic thru five points P, Q, R, M, N, our two conics must coincide and cut C^3 in one sixth point, or O' coincides with O.

THEOREM XIX.

If a line cuts C^3 in three points A, B, C, of which A_1, B_1, C_1, are conjugate points, then AA_1, BB_1, CC_1, are three pairs of opposite vertices of a complete four side whose diagonals AA_1, BB_1, CC_1, cut C^3 in three new points P, Q, R, which form a point triple of C^3.

PROOF.

If AA_1, BB_1, are two pairs of conjugate points of C^3 then CC_1 will be a third pair if line AB cuts A_1B_1 in C and AB_1 cuts A_1B in C_1. The four lines ABC, AB_1C_1, A_1B_1C, A_1BC_1, form a complete quadrilateral and determine a family of conics. The diagonals of the quadrilateral are AA_1, BB_1, CC_1. Let line BB_1 cut CC_1 in A', CC_1 cut AA_1 in B', and AA_1

cut BB_1 in C'. The triangle A', B', C' is self polar with regard to all conics tangent to the sides of the quadrilateral. Line AA_1 cuts C^3 in a third point P, BB_1 cuts C^3 in Q and CC_1 cuts C^3 in R.

One conic K^2 of the family is also tangent to the line PQ. Lines PB' and PA' are conjugate with respect to K^2 since A' is the pole of PB' with respect to all conics of the family. Since PQ is tangent to K^2 the fourth harmonic of PQ with respect to lines PB' and PA' must be a second tangent t from P to K^2. Similarly the fourth harmonic of QP with reference to QA' and QB' is a second tangent t' from Q to K^2. The tangents t and t' must cut the line $A'B' \equiv CC'$ in the fourth harmonic of the point of intersection of PQ and CC_1, with respect to A' and B'. Call R' the point of intersection of t and t'. Since the sides of the two triangles A, B, C_1, and P, Q, R', are tangent to one conic the six vertices lie on a conic K_1^2. * The three points A, B, C_1, form a triple of C^3. (Theorem 16).

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*(If the vertices of two triangles lie on a conic, the sides touch a conic and conversely.)

Since the conic K_1^2 cuts C^3 in three points of a triple the three remaining points of intersection with C^3 must form a point triple. (Theorem 18). Then K_1^2 cuts C^3 in the point which forms the third vertex of the triple with P and Q.

Since P is on the line B'C' of the self polar triangle A', B', C', the pole of PQ must be on a line thru A', and moreover, must be on a line thru B' or in other words, must be on the line $A'B' \equiv CC_1$. Since conic K_1^2 passes thru R' and C_1 on CC_1 and by the above must contain the third vertex of the triple with P and Q, which is also on the line CC_1 then R' is the third point of the desired triple and is therefore a point of C^3. Thus R' coincides with R and the cutting points P, Q, R, of lines AA_1, BB_1, CC_1 with C^3 form a triple of C^3.

In addition it follows that the conjugate points P_1, Q_1, R_1 of P, Q, R must lie on a line. (Theorem 15). P_1 is the point of intersection of tangents to C^3 at A and A_1, Q_1 of the tangents at B and B_1 and R_1 of those at C and C_1. (Theorem 3). Thus we may state the following:

THEOREM XX.

If a line cuts C^3 in three points A, B, C, the tangents to C^3 at these points cut C^3 in three new points which also lie on a line.

THEOREM XXI.

The tangents to C^3 at three points of a triple, P, Q, R, cut C^3 in three new points which lie on a line.

PROOF.

Let line PQ cut C^3 a third time in C. If P_1 is conjugate to P then QP_1 cuts C^3 in R. The tangents to C^3 at P, Q, and C cut C^3 in T_p, T_q, T_c on a line. (Theorem 16). Since Q, P_1, R, are also on a line the tangents to C^3 at these points must intersect C^3 in three collinear points T_q, T_{P_1}, T_R. The points T_p and T_{P_1} coincide since tangents to C^3 at two conjugate points intersect on C^3. Thus on the two lines $T_p T_q T_c$, and $T_q T_{P_1} T_R$, we have two points in common or the lines coincide. It follows that T_R coincides with T_c and the tangents at P, Q, R cut C^3 in collinear points.

THEOREM XXII.

If a line g cuts C^3 in three points A, B, C, and a line g' cuts C^3 in A', B', C', and the first three points are connected with the three last (in any order) the three resulting lines AA', BB', CC' cut C^3 in three new collinear points.

THEOREM XXIII.

Given two triples A_1, B_1, C_1 and A_1', B_1', C_1', the three lines joining the points of the two groups by pairs, A_1A_1', B_1B_1', C_1C_1', cut C^3 in three collinear points.

PROOF.

Let a line cut C^3 in three points A, B, C, whose conjugate points are, A_1, B_1, C_1, and a second line cut C^3 in the points A', B', C' whose conjugate points are A_1', B_1', C_1'. Then by threes these twelve points lie on eight lines. Let the line ABC be d; AB_1C_1 be a: BC_1A_1 be b: CA_1B_1 be c; $A'B'C'$ be d'; $A'B_1'C_1'$ be a'; $B'C_1'A_1'$ be b'; $C'A_1'B_1'$ be c'.

These eight lines are tangent to the same conic. (Theorem 7). From six tangents of the conic may be formed a simple hexagon. We wish to find the point of intersection of the three lines connecting opposite vertices. (Brianchon point). *

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*(The necessary and sufficient condition that six lines, no three of which are concurrent, be lines of a line conic is that the lines joining the three pairs of opposite vertices of any simple hexagon of which the given lines are sides, shall be concurrent.)

In the following notation the six side is represented above and below it are the corresponding lines which must intersect in a point. (ab', ba' shall represent the line thru the intersections of lines a and b' and lines b and a'.)

(a d b a'd'b'
1. (
(AA', BB'; ab', ba'

(b d c b'd'c'
2. (
(BB', CC'; bc', cb'

(c d a c'd'a'
3. (
(CC', AA'; ca', ac'

(b c a b'c'a'
4. (
(A_1A_1', B_1B_1'; ab', ba'

(c a b c'a'b'
5. (
(B_1B_1', C_1C_1'; bc', cb'

(a b c a'b'c'
6. (
(C_1C_1', A_1A_1'; ca', ac'

(a b' c a' b c'
7. (
(ab', ba'; bc', cb'; ca', ac'.

Consider two triangles with sides AA_1, BB_1, CC_1, and $A'A_1'$, $B'B_1'$, $C'C_1'$. From (1) and (4) the line ab', ba' contains the points of intersection of AA_1, BB_1 and $A'A_1'$, $B'B_1'$. (One vertex of each triangle.) From (2) and (5) the line bc', cb' contains the points of intersection of lines BB', CC' and $B_1 B_1'$, $C_1 C_1'$. (Second vertex of each triangle). From (3) and (6) the line ca', ac' contains the points of intersection of CC', AA' and $C_1 C_1'$, $A_1 A_1'$. (Third vertex of each triangle.) By (7) the three lines ab', ba'; bc', cb'; ca', ac'; meet in a point and we have the two triangles with sides AA_1, BB_1, CC_1 and $A'A_1'$, $B'B_1'$, $C'C_1'$ respectively, in perspective. It follows that their corresponding sides intersect on a line.

Let AA' cut the corresponding side $A_1 A_1'$ in A''
BB' cut the corresponding side $B_1 B_1'$ in B''
CC' cut the corresponding side $C_1 C_1'$ in C''.

Then by the above A'', B'', C'' lie on a line. Moreover, by Theorem I, the points A'', B'', C'' lie on C^3.

We thus have lines AA' cutting C^3 in A'',
BB' cutting C^3 in B'',
CC' cutting C^3 in C'', or

AA', BB', CC' cutting C^3 in three collinear points. In the same way $A_1 A_1'$, $B_1 B_1'$, $C_1 C_1'$ cuts C^3 in three collinear points.

In addition to the two preceding theorems a third theorem may be obtained. The notation used above will be retained. If A'' is the point of intersection of AA' and $A_1 A_1'$, and B'' and C'' are obtained similarly, then the conjugate points A_1'', B_1'', C_1'' may be obtained in the following manner by the use of Theorem I. The intersection of AA_1' and $A_1 A'$ will be A_1'', of BB_1' and $B_1 B'$ will be B_1'', and of CC_1' and $C_1 C'$ will be C_1''.

Since A", B", C" lie on a line their conjugate points form a point triple. (Theorem 16). Thus the following theorem may be stated:

THEOREM XXIV.

If a line g cuts C^3 in three points A, B, C and A_1', B_1', C_1' are points of a triple, the connecting lines AA_1', BB_1', CC_1' cut C^3 in three new points which form a point triple.

VI. GENERATION OF C^3 BY MEANS OF A PENCIL OF CONICS AND PROJECTIVE PENCIL OF LINES.

Let A, B, C, D be four points of C^3. If line AB cuts C^3 in C_1' and CD cuts C^3 in C_2', then AC, BD, $C_1'C_2'$, cuts C^3 in three collinear points, (Theorem 22), and also the intersections of AD, BC, $C_1'C_2'$, with C^3 are collinear. Similarly if the intersection of C^3 by AC is B_1', by BD is B_2', by BC is A_1', and by AD is A_2', then B_1', B_2' and the point of intersection of $C_1'C_2'$ with C^3 must lie on a line as also do A_1', A_2' and the intersection of $C_1'C_2'$ with C^3. Thus the lines $B_1'B_2'$, [illegible]', $C_1'C_2'$, and $A_1'A_2'$, pass thru a point Q of C^3 and we have three lines of a pencil thru Q.

The points A, B, C, D determine a pencil of conics. The three line pairs AB, CD; AC, BD; CB, AD; may be thought of as degenerate conics of the pencil. We may set up a projectivity between the pencil of conics and the pencil of lines thru Q in which the lines QA', A_2'; QB', B_2'; QC', C_2'; correspond to the conics CB, AD; AC, BD; AB, CD; respectively.

We wish to show that the locus of the points of intersection of corresponding elements of the two projective forms is a cubic curve. Let any ling g cut the pencil of conics in pairs of points R_1, R_2--- of an involution. * Thus QR_1, QR_2, -- are lines of an involution pencil. Place a conic K^2 thru Q cutting QR_1, QR_2, in N_1, N_2---. The lines $N_1 N_2$ will pass thru a fixed point S and form a pencil of lines each one of which determines a definite conic of the pencil.** The pencil of lines thru S may be taken projective with the pencil thru Q since the pencil S may be taken as a pencil of polars and the corresponding pole with respect to the conics determined. All pencils of polars with respect to a pencil of conics thru four points are projective. (Theorem 9). The two projective pencils S and Q determine a conic K_1^2, thru S and Q which cuts K^2 in three points besides the point Q. Each line thru S determines a conic of our pencil cutting g in some point pair $R_1 R_2$.

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*(The intersections of a straight line by a pencil of conics thru four points form pairs of points in involution.)

**(If conjugate points of an involution on a conic are connected these lines are concurrent.)

If K^2 and K_1^2 intersect at N_1', then line QN_1', corresponds to SN_1', or QN_1', corresponds to the conic of the pencil thru R_1', R_2'. Since this occurs three times we shall have three times on any line g the points of intersection of one of the lines of Q with its corresponding conic. Thus the locus of the points of intersection of corresponding elements of the two projective forms is a C^3.

Is the C^3 thus generated the cubic thru the points which we originally assumed on C^3? (The points $A, B, C, D, C_1', C_2', B_1', B_2', A_1', A_2', Q$)

The point A is on our locus since line QA necessarily cuts the corresponding conic in A. Similarly B, C, D, must lie on the locus. The line QA_1', A_2' by hypothesis cuts the corresponding conic CB, AD in A_1' and A_2'. The same holds true for points B_1', B_2', C_1', C_2'. One conic of the pencil passes thru Q and the line thru Q corresponding to this conic cuts it in Q.

We then have eleven points of the resulting cubic coinciding with the chosen cubic and the two curves coincide throughout since two distinct cubics cannot have more than nine points in common.

It follows that if A, B, C, D, are any four points of C^3, each conic of the pencil determined by A, B, C, D cuts C^3 in two other points R_1, R_2. The lines $R_1 R_2$ will pass thru a fixed point Q of C^3 and form a pencil of lines Q which is projective with the pencil of conics. The point Q in reference to the four points A, B, C, D will be called the opposite point.

Choose four points A, B, C, Q of C^3. Let the cubic be cut by line BC in A_1', by CA in B_1', by AB in C_1', by QA_1' in A_2', by QB_1' in B_2' and by QC_1' in C_2'. Since A, B, C_1' are collinear and B_1', B_2', Q are collinear then lines AB_1', BB_2', QC_1', intersect C^3 in three collinear points. (Theorem 22). If the first is C, and the third is C_2', then BB_2' intersects C^3 in a point of line CC_2'.

The points B, C, A_1' are collinear and the points C_1', C_2', Q are collinear and it follows that lines BC_1', CC_2', QA_1', intersect C^3 in collinear points of which the first and third are A and A_2'. Thus CC_2' cuts AA_2' in a point of C^3. Since CC_2' contains only one additional point D on C^3 the lines AA_2' and BB_2' intersect C^3 in D. The point Q is the opposite point of the four points A, B, C, D, and each line of the pencil thru Q will cut C^3 in two points R_1, R_2 which lie on a conic of the pencil thru A, B, C, D. Therefore we may state:

THEOREM XXV.

If three points A, B, C are taken on C^3 and lines are drawn thru a fourth point Q each cutting C^3 in two additional points

R_1, R_2, the conics determined by A, B, C, R_1, R_2 will all pass thru a fourth point D of C^3 forming a pencil of conics projective with the pencil of lines thru Q.

One can pass from the generation of C^3 by means of a pencil of conics and projective pencil of lines to the generation by means of two projective half perspective line involutions if the four fixed points of the pencil be chosen suitably. Let the four fixed points be A, B, C, P and the corresponding opposite point be P_1. The line corresponding to that conic of the pencil which goes thru P_1 must be a line in which two cutting points with C^3 have coincided. (In general the lines thru P_1 cut C^3 in P_1, R_1, R_2. In this case one of the pair R_1, R_2 has moved to P_1.) This line is then a tangent to C^3 in P_1. Call T the third cutting point of this tangent with C^3. Then T also lies on the conic thru A, B, C, P, P_1. Similarly if we choose A, B, C, P_1 as the points determining our pencil of conics and P as the corresponding opposite point, the conic thru A, B, C, P_1, P cuts C^3 for a sixth time in the point where the tangent to C^3 at P cuts C^3 for a third time. If this is to hold it follows that P and P_1 must be conjugate points of C^3 as the tangents to C^3 at P and P_1 must intersect in a point T of C^3. Thus if P and P_1

are conjugate points of C^3 the pencil A, B, C, P, will have P_1 as ppposite point and the pencil A, B, C, P_1 will have P as opposite point. The points P, P_1, T form a point triple of C^3; for, to obtain the third point of a triple with reference to any two points P and T, connect T with the conjugate of P (or P_1) and the third point of intersection of P_1 T is the desired point. This point however, is P_1 itself.

Consider the family of conics thru A, B, C, P and the projective pencil of lines thru P_1. If any line thru P_1 cuts the corresponding conic in R_1, R_2, these points are points of C^3. Let lines PR_1, PR_2, cut C^3 in N_2 and N_1 respectively. Thinking of P as opposite point in reference to A, B, C, P_1 the points A, B, C, P_1, R_1, N_2 will lie on a conic with $PR_1 N_2$ as the corresponding line of the pencil thru P. Likewise A, B, C, P_1, R_2, N_1, will lie on a conic with $PR_2 N_1$ as corresponding line. Since P, R_1, N_2, lie on a line and P, R_2, N_1, on a second line then the lines PP, $R_1 R_2$, $N_1 N_2$, cut C^3 in three collinear points. The line PP cuts C^3 in T, and the line $R_1 R_2$ cuts C^3 in P_1, so the line $N_1 N_2$ must pass thru the third point of intersection of the line TP_1 and C^3. Since this point is P_1 itself, P_1, N_1, N_2, are collinear. Therefore the conic thru A, B, C, P, N_1, N_2

will correspond to the line $P_1N_1N_2$ of the pencil thru P_1.

The four lines $P_1R_1R_2$, $P_1N_1N_2$, PR_1N_2, PR_2N_1 form a cpmplete quadrilateral with PP_1, R_1N_1, R_2N_2, as pairs of opposite vertices. Take Z_1, Z_2, T_1', T_2' such that the following harmonic sets are formed.

$H(R_1R_2, P_1Z_1)$, $H(N_1N_2, P_1Z_2)$, $H(R_1N_2, PT_1')$ $H(R_2N_1, PT_2')$. The points Z_1, Z_2, P lie on a line as do the points T_1', T_2', P_1. The point of intersection M of lines R_1N_1 and R_2N_2 will coincide with that of Z_1Z_2 and $T_1'T_2'$. * As the conics vary the points R_1, R_2, N_1, N_2, Z_1, Z_2, T_1', T_2', will vary but line Z_1Z_2 will continually pass thru P and line $T_1'T_2'$ thru P_1. The polars of P_1 with respect to the conics of the pencil must pass thru Z_1 or the fourth harmonic point of P_1 on each line $P_1R_1R_2$ with respect to R_1 and R_2. The pencil of polars is projective with the pencil of conics, moreover the pencil of lines thru P_1 is projective with the conics. Therefore, we have the pencil of polars with respect to the point P_1 projective with the pencil thru P_1.

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*(The two sides of a complete quadrangle which meet in a diagonal point are harmonic conjugates with regard to the two sides of the diagonal triangle which meet in this point.)

Since each polar meets its corresponding line in Z_1 the point Z_1 will describe a conic on which Z_2 will also lie. The variable chord $Z_1 Z_2$ of this conic passes thru P. Thus $P_1 Z_1$ or $P_1 R_1$ and $P_1 Z_2$ or $P_1 N_2$ are conjugate lines of an involution. #
Similarly PT_1' or PR_1 and PT_2' or PR_2 will be conjugate lines of an involution.

It must be shown that these two involutions are projective. In Section I it was shown that if $R_2 N_2$, PP_1, $R_1 N_1$, are pairs of conjugate points of C^3 the fourth harmonics of the intersections of line PP_1 and the line $R_2 N_2$ with respect to R_2 and N_2, and the line $R_1 N_1$ with respect to R_1 and N_1, --- lie on a line. Therefore, the point which we have called M will, as the conics vary, lie on a line. This gives a perspectivity between the pencil of lines $P_1 T_1' T_2'$ and the pencil $PZ_1 Z_2$. In section I it was also shown that two involutions P and P_1 are projective if so situated that the pencils of lines formed thru P and P_1 and composed of the fourth harmonic lines to PP_1 and $P_1 P$ with regard to the pairs of conjugate lines of the respective involutions, are projective.

It remains to show that the involutions P and P_1 are also in half perspective.

The conic thru A, B, C, P, P_1, T belongs to both pencils and in the one case we obtain the line pair $P_1 T$ and $P_1 P$ and in the other case the line pair PT and PP_1 . Thus corresponding line pairs of the two involutions have one line PP_1 of the one corresponding to P_1P of the other, or the two involutions are not only projective but also in half perspective.

The generation of C^3 by means of a pencil of conics projective with a pencil of lines has been reduced to the genaration by means of two projective half perspective line involutions.

As a result of the method of generation of C^3 by means of a pencil of conics and projective pencil of lines we may state:

THEOREM XXVI

If any six points of C^3 lie on a conic their six conjugate points also lie on a conic.

PROOF.

If a conic cuts C^3 in A, B, C, D, E, F the line EF cuts C^3 in the opposite point Q of C^3 with reference to the four points A, B, C, D which determine a pencil of conics. If E_1 F_1 are conjugate points of E and F then line EF cuts E_1F_1 in a point of C^3 . (Theorem I). Then E_1F_1 cuts C^3 in Q, and in general it will follow that if there is a conic thru A, B, C, D, E, F there is also a conic thru A, B, C, D, E_1, F_1 . Similarly there would be one thru A, B, C_1, D_1, E_1, F_1 and finally thru A_1, B_1, C_1, D_1, E_1, F_1 .

THEOREM XXVII.

If a conic cuts C^3 in six points A, B, C, D, E, F, the three chords AB, CD, EF, cuts C^3 in three collinear points.

PROOF.

If A, B, C, D are taken as the four points determining a pencil of conics, then line EF passes thru the opposite point Q of C^3. Moreover as a degenerate conic of the pencil we have the line pair AB and CD, the third points of intersection of which with C^3 must lie on a line thru Q. Thus AB, CD and EF cut C^3 in three collinear points.

VII. CONSTRUCTION OF C^3 THRU NINE GIVEN POINTS BY MEANS OF A PENCIL OF CONICS AND PROJECTIVE PENCIL OF LINES.

The number of points used in the generation of C^3 which may be chosen arbitrarily is to be determined. Consider the four fixed points A, B, C, D and opposite point Q. The points A, B, C, D may at least be chosen at will (except that no three are to be chosen on a line). In reference to the two cutting points of a line thru Q with its corresponding conic only one may be taken at random as the second is definitely determined by the first.

If we take three of these points the projectivity of the pencil of lines and conics is determined for any Q. We will then take four such points E, F, G, H and consider the determination of Q such that the lines QE, QF, QG, QH, shall correspond respectively to the conics thru A, B, C, D, E; A, B, C, D, F; A, B, C, D, G; A, B, C, D, H. The above lines and conics may be represented by Q(EFGH) and (ABCD) (EFGH). The projectivity (ABCD) (EFGH) $\overline{\wedge}$ Q(EFGH) limits

Q to a conic thru E, F, G, H which determines a definite double ratio.* This conic K^2 can be constructed. The value of the double ratio is given by the four conics (ABCD) (EFGH) and may be determined by the tangents to the conics at one of the fixed points or by the double ratio of the range of points formed by the remaining four points of intersection with the conics by a transversal thru one of the fixed points.

Thus thru eight points infinitely many cubic curves may be generated since Q may be any point of the conic K^2. When Q is chosen the projectivity is fixed and C^3 thus determined.

Moreover all cubics thru the eight independent points will also pass thru one definite ninth point O. Take the conic K^2 and a chosen position of Q resulting in a definite curve C_1^3. The conic K^2 cuts C_1^3 in a sixth point O in addition to the five points E, F, G, H, Q, and therefore (ABCD) (EFGHO) $\overline{\wedge}$ Q(EFGHO). Take a second position or Q_1 on K^2 thus determining a second cubic C_2^3.

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*(The double ratio of a pencil formed by joining any four fixed points on a conic to a variable point on the conic is constant.)

Since $Q(EFGHO) \bar{\wedge} Q_1 (EFGHO)$ it follows that $Q_1 (EFGHO) \bar{\wedge} (ABCD)$ $(EFGHO)$ and since C_2^3 is generated by intersections of corresponding lines and conics of the two projective pencils, the point O is also on C_2^3. The above statements hold for any choice of Q_1 on K^2 and as a result we may state the following theorem:

THEOREM XXVIII.

All cubics generated thru eight chosen points will pass thru a ninth dependent point.

These nine points form a group of nine associated points in which any one of the nine may be obtained from the remaining eight.

However, thru nine independent points there is one and only one cubic curve. * Let A,B,C,D,E,F,G,H,I be nine arbitrarily chosen independent points. (Independent points are such that no three lie on a line, no six on a conic and the nine do not form a group of associated points.)

Take four conics (ABCD) (EFGH) and determine a conic K^2 thru E,F,G,H, such that the double ratio of the lines connecting any point on K^2 with E,F,G,H, is the double ratio of the four conics. Take four conics (ABCD) (EFGI) and determine similarly K_1^2 thru E,F,G,I. Since K^2 and K_1^2 pass thru E,F,G they have one other point Q in common. Then $(ABCD)\ (EFGHI) \bar{\wedge} Q(EFGHI)$ and Q is the focus of the generating pencil of lines determining a cubic thru the

*(Construction given by Chasles in the Comptes Rendus 1853 Vol. 36 Page 951)

the nine independent points.

VIII. A SECOND METHOD OF GENERATING C^3 BY MEANS OF PENCILS OF CONICS AND PENCILS OF LINES.

Consider a pencil of conics thru A, B, C, D and a projective pencil of lines thru Q. Let conic X^2 correspond to line x and let R_1, R_2 be the intersections of x with X^2. Any line l thru D cuts the conics in a range of points N_1, N_2 ---N_i which is projective with the pencil of conics and thus with the pencil of lines Q. (Theorems 9 and 10).

Let X_0^2 be the conic thru A, B, C, D, Q cutting l in N_0 and the variable line x in R'. The three conics X^2, X_0^2 and the degenerate conic formed by the line pair l, x have the point D in common. Similarly in common with X^2 and X_0^2 are the points A, B, C, with X^2 and (l, x) the points N, R_1, R_2, with X_0^2 and (l, x) the points N_0, Q, R'. It follows that the lines AB, AC, BC, NR_1, NR_2, x, N_0Q, N_0R', are tangent to a conic K^2. *

As x varies our conic K^2 varies but K^2 will continually have the four fixed lines AB, AC, BC, QN_0, as tangents.

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*(If three conics have a point in common, and by pairs they have in addition three other common points, each pair giving a triangle. The nine sides of the three triangles thus obtained are tangent to the same conic. Synthetisehe Geometrie by Jacob Steiner. Page 242).

Thus K^2 describes a pencil of conics. From a fixed point Q on one of the fixed tangents of the family there is always a second tangent x to each conic of the family.

The pencil of tangents Q(x) is projective with the pencil of conics, for as a dual of (theorem 10) we have: Given a pencil of conics tangent to four lines, a point on one of the fixed tangents sends a second tangent to each conic forming a pencil of lines projective with the pencil of conics. The pencil of points N_1, N_2-- is projective with the pencil of lines Q(x) and thus with the pencil of conics.

This gives a new method of generating C^3. Take a family of conics with four common tangents and a projective range of points N on a line l. From a fixed point Q on one of the four common tangents draw a second tangent x to each conic of the family. From the point N_i corresponding to conic K_i^2 draw two tangents cutting the line x in R_1, R_2. The locus of the points R_1, R_2--- is the C^3 which passes thru Q and the vertices of the triangle formed by the three fixed tangents other than the one on which Q has been chosen. Call the vertices of the triangle A, B, C. Since R_1 or R_2 must coincide with A, B, C, and Q respectively for some positions of Q(x) and since for every position of R_1 or R_2 the lines NR_1 and NR_2 are tangent to K_i^2, then it must follow that the cubic so generated will pass thru A, B, C, and Q.

Since the line l thru D may be chosen arbitrarily in this method QN_o or the fourth tangent may be chosen arbitrarily.

IX. A SECOND METHOD OF GENERATING C^3 THRU NINE INDEPENDENT POINTS.

Let the nine points be A, B, C, Q, E, F, G, H, I. Thru Q draw any line g. Determine five conics having the four common tangents AB, AC, BC, g and in addition the one tangent QE, QF, QG, QH, QI, respectively. From E, F, G, H, I draw to each conic the second tangent t_E, t_F, t_G, t_H, t_I. This determines a line l which cuts the five tangents in the points N_E, N_F, N_G, N_H, N_I, such that the range of points N_E, N_F, N_G, N_H, N_I, on l is projective with the pencil of lines Q(E, F, G, H, I). Determine the conic C_1^2 tangent to t_E, t_F, t_G, t_H such that the cross ratio determined by the above tangents on any fifth tangent to C_1^2 is the same as the cross ratio of the four lines QE, QF, QG, QH. * Determine a second conic C_2^2 tangent to t_E, t_F, t_G, t_I such that the cross ratio determined by the above tangents on any fifth tangent to C_2^2 is the same as the cross ratio of the four lines QE, QF, QG, QI. By the construction the conics C_1^2 and C_2^2 have three common tangents. The fourth common tangent will be our line l.

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*(Any four fixed tangents to a conic determine a range of constant anharmonic ratio on all other tangents to the conic.)

Each ppint N of the range on l corresponds then to a definite line x of the pencil thru Q, and to a definite conic X^2 of the family and having x as a fifth tangent.

From N drawn tangents to X^2 cutting x in the pair of points R_1, R_2. These points generate C^3. By the construction the tangents thru N_E, N_F, N_G, N_H, N_I, cut x in E, F, G, H, I respectively and moreover these were chosen as corresponding lines and points. (That is $QE \equiv x$ corresponds to N_E) Thus the points E, F, G, H, I lie on C^3 and from the discussion in the preceding Section A, B, C, and Q must also lie on C^3.

This method of generating a cubic thru nine independent points is given without demonstration by Chasles in Liouvilles Journal Vol. 19, page 366.

X. GENERATION OF C^3 BY MEANS OF TWO PROJECTIVE PENCILS OF CONICS.

The generation of C^3 thru two line involutions which are projective and in half perspective may be generalized by regarding a line involution as a special case of a pencil of conics in which each conic has degenerated into a pair of lines. A pencil of conics is determined by the four points of intersection of two conics. Consider for the determination of a pencil of conics the two pairs of lines aa_1, bb_1 in which Q is the point of intersection of a and a_1 as well as of b and b_1.

In this case the four fixed points which determine the pencil have coincided in Q. Every other conic of the pencil must have with the conic A^2 or $[a, a_1]$ and B^2 or $[b, b_1]$ only the point Q in common and must therefore degenerate into a pair of lines thru Q. Moreover, the pencil of conics must have the characteristic that the conics cut any line in pairs of conjugate points of an involution. Thus our pencil of degenerate conics becomes a line involution.

Take two projective pencils of conics thru the points A, B, C, D, and A_1, B_1, C_1, D_1, respectively. Except in special cases we would have a fourth degree curve generated by the points of intersection of corresponding conics since the points of corresponding pairs of the two involutions formed on any line g will in general coincide four times (Section III). However, the pencils of conics can be so formed that the points of intersection break up into a straight line and a curve. In other words one of the four points on any line g will describe a straight line l and the fourth degree curve breaks up into a third degree curve and a straight line. This was obtained in the case of the two projective line involutions by means of the half perspective position.

Two projective pencils of conics thru A, B, C, D, and A_1, B_1, C_1, D_1, which form on one definite line l the same point involution will satisfy the requirement. Then of the four points in which any line g cuts the locus the point of intersection of l and g will be excluded and the remaining three will lie on a cubic curve.

To generate a C^3 passing thru nine independent points $A, B, C, D, A_1, B_1, C_1, E, F$, consider the conic thru A, B, C, D, E, and the pencil of conics thru A_1, B_1, C_1, E. Each conic of the pencil cuts the fixed conic in three points other than the point E. The triangles formed thru the remaining three points of intersection of each conic of the pencil with the fixed conic are tangent to a conic K^2. * The sides of the triangle $A_1 B_1 C_1$ are tangent to K^2.. Take the conic thru A, B, C, D, F, and the pencil thru A_1, B_1, C_1, F. Similarly we have a conic K_1^2 with the sides of the group of triangles thus formed as tangents. Among these triangles is again the triangle $A_1 B_1 C_1$. Since K^2 and K_1^2 have three common tangents they must also have a fourth real tangent l.

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*(Steiner's Synthetische Geometrie, P. 242).

To show that the line l possesses the characteristic that its points of intersection with the conic thru A, B, C, D, E, lie on a conic thru A_1, B_1, C_1, E, and its points of intersection with the conic thru A, B, C, D, F, lie on a conic thru A_1, B_1, C_1, F, we will assume that l cuts the conic thru A, B, C, D, E, in points $N_1 N_2$. Pass a conic of the pencil determined by A_1, B_1, C_1, E, thru the point N_1. It will cut conic thru A, B, C, D, E, in two remaining points P_1, P_2. The sides of the triangle $N_1 P_1 P_2$ are tangent to K^2. From the point N_1 there are two tangents $N_1 P_1$ and $N_1 P_2$ to K^2 but by hypothesis $N_1 N_2$ or l is tangent to K^2. Thus N_2 must coincide with either P_1 or P_2 and the points N_1 and N_2 both lie on conic C^2 thru A_1, B_1, C_1, E. Similarly if l cuts conic A, B, C, D, F in N_1', N_2' the points A_1, B_1, C_1, F, N_1', N_2' lie on a conic C_1^2. C^2 and C_1^2 intersect in A_1, B_1, C_1 and a fourth point D_1.

The two pencils of conics thru A, B, C, D, and A_1, B_1, C_1, D_1 form the same point involution on l since an involution is determined by two pairs of conjugate points and the two involutions on l each have $N_1 N_2$, $N_1' N_2'$ as pairs of conjugate points. Moreover, the involution on l determines the projectivity between

The pencils of conics. Take any point P on l. The pencil of polars of P with respect to the conics thru A, B, C, D, will intersect l in the fourth harmonics of P with respect to N_1, N_2 ---. The pencil of polars of P with respect to the conics thru A_1, B_1, C_1, D_1 will also intersect l in the fourth harmonics of P with respect to the same series of points N_1, N_2 ---. Thus the two pencils of polars are projective and as a result the two pencils of conics are projective. The polar of P with regard to conic A, B, C, D, N_1, N_2 will correspond to the polar of P with regard to conic A_1, B_1, C_1, D_1, N_1, N_2 or conic A, B, C, D, N_1, N_2 corresponds to conic A_1, B_1, C_1, D_1, N_1, N_2.

The remaining points of intersection of corresponding conics will generate a C^3 passing thru the nine chosen points. The above discussion insures that E and F are points of intersection of corresponding conics and thus on C^3. It remains to show that any one of the other original points, for example B, is on the C^3 thus generated. One conic of the pencil A_1, B_1, C_1, D_1, will pass thru B and thus intersect in B the corresponding conic of the second pencil since all conics of the second pencil pass thru B. Similarly it may be shown that the other chosen points lie on C^3.

Thus given nine independent points, two projective pencils of conics have been determined which generate a cubic thru the given points. *

XI. SECOND METHOD OF GENERATING C^3 BY MEANS OF TWO PROJECTIVE PENCILS OF CONICS. **

Consider the generation of a cubic thru the nine points $A, B, C, A_1, B_1, C_1, E, F, G$. Two conics thru A, B, C, E, F, and A_1, B_1, C_1, E, F, have in addition to the points E and F, two points P, P_1 in common. Points P, P_1 determine a line g. The pencils of conics A, B, C, G, and A_1, B_1, C_1, G, determine on the line g two point involutions. Let the common pair of points of the two involutions be Q, Q_1. The pair of elements belonging to two involutions when both involutions are on the same form, may be found by construction.

The two pairs of points PP_1, QQ_1, determine on g a new point involution the pairs of conjugate points of which may be represented by $R_1 R_1'$, $R_2 R_2'$---.

The conics A, B, C, R_i, R_i', each pass thru a fixed point D and the conics A_1, B_1, C_1, R_i, R_i', a fixed point D_1.

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*(Chasles in Comptes Rendus Vol. 36, 1853 page 943).

**(Chasles in Comptes Rendus Vol. 41 Dec. 1855).

Any two conics thru A, B, C, R_1, R_1', and A, B, C, R_2, R_2', intersect in a point D. The family of conics thru A, B, C, D determine on g a line involution of which R_1, R_1'; R_2, R_2', are pairs of conjugate points. Since two pairs of conjugate points determine an involution the pencil thru A, B, C, D, determines on g the involution in which we are interested. Similarly the pencil thru A_1, B_1, C_1, D_1, may be determined thru the third point of intersection of two conics A_1, B_1, C_1, R_1, R_1', and A_1, B_1, C_1, R_2, R_2'.

By construction the two pencils of conics A, B, C, D, and A_1, B_1, C_1, D_1, determine on g the same involution and this involution fixes the projectivity between the two pencils. (Proved in preceding section).

The locus of the remaining points of intersection of corresponding conics is therefore a cubic. The points A, B, C, A_1, B_1, C_1 may be shown to lie on C^3 by the method used in the section preceding. By the choice of $R_1 R_1'$; $R_2 R_2'$; the points P, P_1 are conjugate points of the involution and a conic of the one family thru P, P_1 cuts the corresponding conic of the second family also thru P, P_1 in the points E and F. Thus points E and F are on C^3. Since QQ_1, are also conjugate points of the involution on g it follows that G is a point of intersection of corresponding conics.

We may then conclude that the C^3 thus generated passes thru the chosen nine points.

XII. REDUCTION OF CERTAIN FURTHER METHODS OF GENERATING C^3 TO METHODS PREVIOUSLY DISCUSSED.

(A) If two three points have a side and vertex x on that side in common, and the two remaining sides in each three point pass respectively thru four fixed points A_1, A_2, B_1, B_2, and in addition the two remaining vertices of each three points lie respectively on four fixed lines a_1, a_2, b_1, b_2, then the point X describes a cubic curve. * This method of generation of the cubic may be reduced to the generation by means of a pencil of lines, and projective pencil of Conics. **

Given four points A_1, A_2, B_1, B_2 and four lines a_1, a_2, b_1, b_2, consider the two triangles with vertices S, R_1, N_1 and S, R_2, N_2, so placed that R_1 is on line a_1, R_2 on a_2, N_1 is the point of intersection of lines $A_1 R_1$ and b_1, N_2 is the point of intersection of $A_2 R_2$ and b_2, and finally S is the common point of intersection of lines $B_1 R_1$, $B_2 R_2$, $N_1 N_2$.

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*(Grassmann's Werke II)

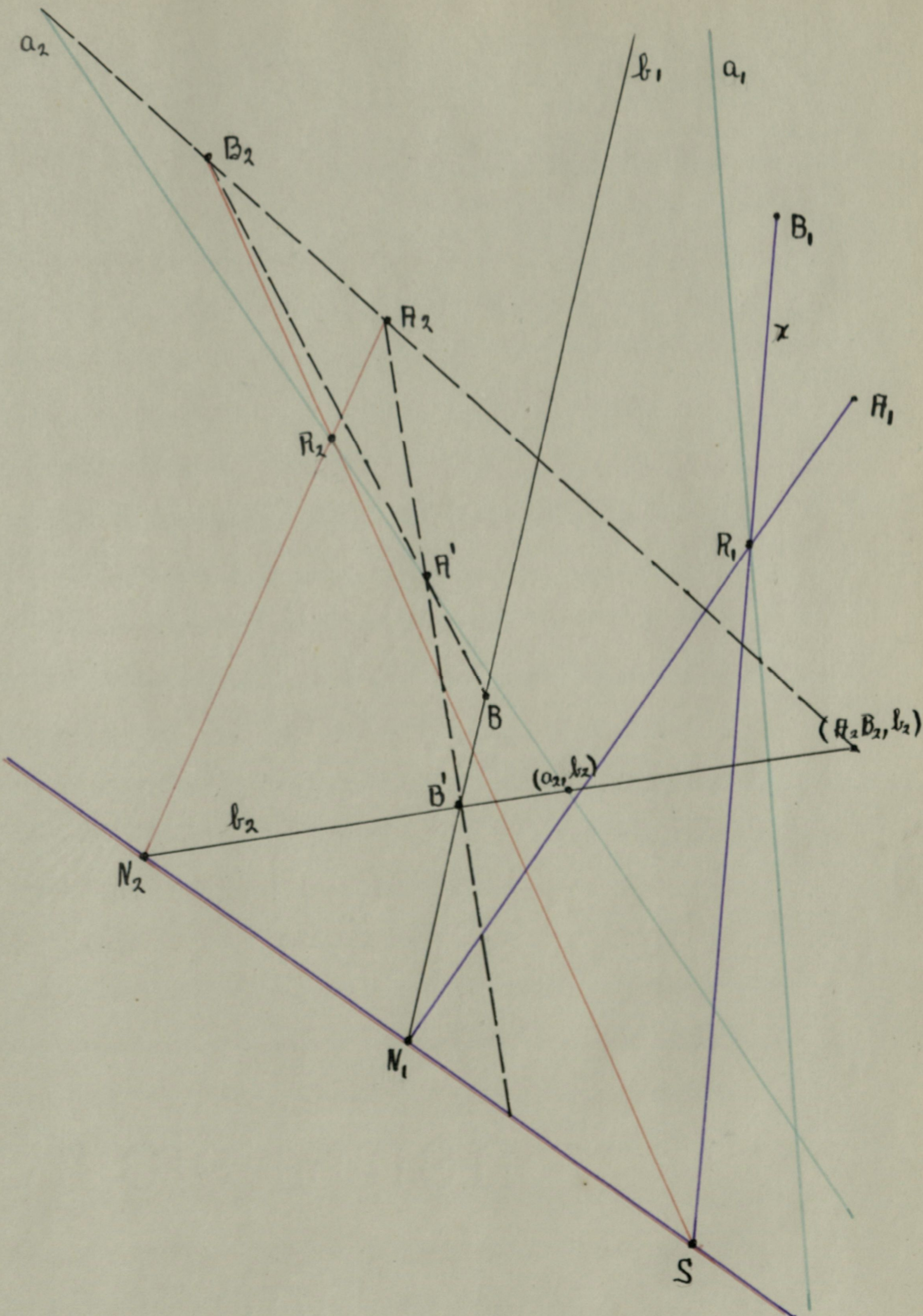
**(Crelle Journal 104, page 62).

By the above definition the point S describes a cubic curve.

Hold line $A_1 R_1 N_1$ fixed and let R_2 describe a_2 and N_2 describe b_2. These two points ranges are porjective since line $R_2 N_2$ always passes thru A_2. Thus $B_2 R_2$ and $N_1 N_2$, will form two projective pencils of lines with vertices B_2 and N_1. S will lie on a conic X^2 thru B_2 and N_1. There will be two points of the locus where conic X^2 cuts the line $B_1 R_1$ or x.

Let R_1 move on a_1 and x will form a pencil of lines with B_1 as vertex. As R_1 varies so does N_1 and for each position we have a new conic X_i^2.

It may be shown that we have a pencil of conics thus formed thru four fixed points. B_2 is a fixed point of the pencil. A_2, R_2, N_2 lie on a line and when this line passes thru the point of intersection of a_2 and b_2 then R_2 and N_2 will coincide and lines $B_2 R_2$ and $N_1 N_2$ will intersect at the point of intersection of a_2 and b_2. Let (a_2, b_2) represent the point of intersection of a_2 and b_2, the second fixed point thru which in each case the conic X_i^2 must pass. When R_2 is the point of intersection of a_2 and line $A_2 B_2$, then N_2 is the point of intersection of b_2 and $A_2 B_2$, and the lines $B_2 R_2$ and $N_1 N_2$ intersect at the point of intersection of b_2 with $A_2 B_2$, which gives us a third fixed point of the pencil. In each position of R_1 and N_1, as



a2
b1
a1
B2
B1
A2
x
A1
R2
R1
A'
B
(A2B2, b2)
(a2, b2)
B'
b2
N2
N1
S

as R_2 and N_2 vary, line $N_1 N_2$ will at one time coincide with b_1. This occurs when N_2 is the point of intersection of b_1 and b_2 and R_2 is the point of intersection of the line thru A_2 and (b_1, b_2) and the line a_2. Thus line $B_2 R_2$ cuts b_1 in a point of X^2 when R_2 is the same fixed point in each case.

Let B' represent the point (b_1, b_2), A' the point of intersection of lines $A_2 B'$ and a_2, and B the cutting point of $B_2 A'$ and b_1. The pencil of conics X^2 will be determined by the four fixed points B_2, (a_2, b_2), $(A_2 B_2, b_2)$, B.

The line b_1 cuts each conic of the pencil X^2 in a fixed point B and the variable point N_1. (N_1 is the vertex of one of the two projective pencils of lines determining X^2). As N_1 describes b_1 we have a range of points projective with the pencil of conics. (Theorem 10) This range of points N_1 is projective with the range R_1 on a_1 since the line $R_1 N_1$ always passes thru A_1, and therefore projective with the pencil of lines $B_1 R_1$ or x.

The pencil of conics X^2 is thus projective with the pencil of lines x and the point S is the locus of points of intersection of corresponding elements of a pencil of lines and projective pencil of conics.

(B) The locus of a point of which the connecting lines with three fixed points so cuts three fixed lines that the three points of intersection lie on a line, is a cubic curve. *

*(Grassmann's Werke II)

This definition may be reduced to the locus of a point which is the vertex of an involution pencil, conjugate lines of which pass thru three pairs of independent points. * *

Take any three points A_1, B_1, C_1 and any three lines a, b, c, Let A be the point of intersection of b and c, B of a and c, and C of a and b. The problem is to find the point R such that line RA_1 cuts a in A', RB_1 cuts b in B', RC_1 cuts c in C', and the points A', B', C' shall lie on a line x.

In this case a, b, c, x will form a complete quadrilateral with AA', BB', CC', as the three pairs of opposite edges and R may be taken as the vertex of an involution pencil conjugate lines of which are RA, RA'; RB, RB'; RC.RC'. From the construction it then follows that RA, RA_1'; RB, RB_1'; RC, RC_1'; are conjugate lines of an involution pencil and the above definition may in this way be reduced to the definition of the cubic curve which was considered in Section I.

(C). A third method of generating a cubic given by Grassmann might be stated. If the sides of a variable simple quadrilateral and a diagonal of the same pass respectively thru five fixed points and the two vertices not on the given diagonal lie in turn on two fixed lines, then the locus of the two vertices on the given diagonal is a cubic curve. This method may also be shown to reduce to the generation of the cubic by means of a pencil of conics and projective pencil of lines.

* Cayley, Liouville Journal de Mathematiques 9. P. 287.
* Crelle Journal 104 P. 62.